Abelian regular coverings of the quaternion hypermap

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Abstract

A hypermap is an embedding of a connected hypergraph into an orientable closed surface. A covering between hypermaps is a homomorphism between the embedded hypergraphs which extends to an orientation-preserving covering of the supporting surfaces. A covering of a hypermap onto itself is an automorphism, and a hypermap is regular if its automorphism group acts transitively on the brins. Depending on the algebraic theory of regular hypermaps and hypermap operations, the abelian regular coverings over the quaternion hypermap are investigated. We define normalized multicyclic coverings between regular hypermaps, generalizing almost totally branched coverings studied in [K. Hu, R. Nedela, N.-E Wang, Branched cyclic regular coverings over platonic maps, European J. Combin. 36 (2014) 531–549]. It is shown that the covering transformation group of a normalized multicyclic covering is a nilpotent group of bounded class. As an application the abelian normalized bicyclic coverings over the quaternion hypemap are classified. In particular, those coverings which possess various level

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of external symmetry or fulfil certain smoothness conditions are explicitly determined.

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1 Introduction

A *hypergraph* is a generalization of a graph, in which every hyperedge may be incident with any number of hypervertices. A *hypermap* is an embedding of a hypergraph into an orientable surface without boundary. In the case when a graph is embedded it is often called a *map*. The Walsh map $W(\mathcal{H})$ of a hypermap $\mathcal{H}$ is a bipartite map on the same surface, with its black and white vertices representing the hypervertices and hyperedges, and edges representing incidence. In this sense the underlying graph of the Walsh map $W(\mathcal{H})$ will be called the *underlying bipartite graph* of the hypermap $\mathcal{H}$.

An automorphism of a hypermap $\mathcal{H}$ is a permutation of its brins, the incident pairs of hypervertices and hyperedges, which preserve the underlying hypergraph and extends to an orientation-preserving self-homeomorphism of the supporting surface. The set of automorphisms of $\mathcal{H}$ forms the automorphism group $\mathrm{Aut}(\mathcal{H})$ of $\mathcal{H}$ under composition. It is well known that $\mathrm{Aut}(\mathcal{H})$ acts semi-regularly on the brins. In the case when this action is transitive, and hence regular, the hypermap $\mathcal{H}$ is called *regular* as well.

Belyĭ's theorem establishes a correspondence between compact Riemann surfaces S which are definable over the number field $\bar{\mathbb{Q}}$ and meromorphic functions $\beta : S \to \Sigma$ which have at most three critical points [1]. The trivial hypermap on the Riemann sphere Σ consisting of a single incident triple of hypervertex, hyperedge and hyperface lifts along β to a hypermap on S. Through the correspondence the absolute Galois group $\mathbf{G} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts naturally on the Riemann surfaces, and this induces a faithful action on the hypermaps [13]. It is remarkable that this action remains faithful when restricted to regular hypermaps [11].

The classification problem of regular hypermaps has become important in this area of research. It has been studied by imposing certain conditions on the supporting surfaces, on the underlying bipartite graphs or on the automorphism groups; see [3, 4, 7, 9, 10, 16, 17, 22, 25, 35] and references therein. On the other hand, the covering techniques, employed by Coxeter-Moser [8], Biggs [2], Gross [12] and others [5, 14, 15, 20, 26, 21, 20, 20, 23, 31, 32, 33, 34] as a tool to investigate certain extensions of the polyhedral groups or regular coverings of symmetrical graphs and maps, extend naturally

to classify regular covering of a given hypermap [26, 28].

In this paper depending on the algebraic theory of regular hypermaps and hypermap operations developed in [6] and [24] we employ the covering techniques to investigate coverings between regular hypermaps. More specifically, in Section 4 we extend *almost totally branched coverings* between regular maps studied in [14, 15] to a broader category and define *normalized multicyclic coverings* between regular hypermaps. By definition the covering transformation group of such a covering is a multicyclic group, a product of several cyclic groups. We show in Theorem 7 that it is nilpotent of bounded class. A complete classification of the covering transformation group of a normalized bicyclic covering is given in Theorem 8.

As an application of our theory we present in Section 5 a classification of abelian normalized bicyclic coverings of the quaternion hypermap, the unique regular hypermap with an automorphism group isomorphic to the quaternion group; see Theorem 9. Explicit descriptions are given in Section 6 for those coverings which possess various level of external symmetry (see Theorem 11, 12, 13, 15) and in Section 7 for those coverings which fulfil certain smoothness conditions (see Theorem 20, 21, 22).

2 Algebraic hypermaps

A hypermap $\mathcal{H}$ may be regarded as a transitive permutation representation $\phi : F_2 \to \langle \rho, \lambda \rangle$ of the free group of rank two

$$F_2 = \langle X, Y \mid - \rangle = \langle X, Y, Z \mid XYZ = 1 \rangle$$

onto a group (the monodromy group $\mathrm{Mon}(\mathcal{H})$ of $\mathcal{H}$) generated by two permutations ρ and λ of a non-empty finite set Φ representing its brins. The hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ are the orbits in Φ of the cyclic subgroups $\langle X \rangle$, $\langle Y \rangle$ and $\langle Z \rangle$ in F_2, with incidence given by non-empty intersection.

The stabilizer in F_2 of a brin $\nu \in \Phi$ is a subgroup H of finite index in F_2, uniquely determined up to conjugacy. It will be called the *hypermap subgroup* associated with $\mathcal{H}$. The automorphism group $\mathrm{Aut}(\mathcal{H})$ of $\mathcal{H}$ is the centralizer of $\mathrm{Mon}(\mathcal{H})$ in $\mathrm{Sym}(\Phi)$. We have $\mathrm{Aut}(\mathcal{H}) \cong N/H$ where $N = N_{F_2}(H)$ is the normalizer of H in F_2. Due to the transitivity of $\mathrm{Mon}(\mathcal{H})$, the group $\mathrm{Aut}(\mathcal{H})$ is semi-regular on Φ. In the case when $\mathrm{Aut}(\mathcal{H})$ is transitive, and hence regular, the hypermap $\mathcal{H}$ is called *regular*. This is equivalent to that H is normal in F_2, in which case we have $\mathrm{Aut}(\mathcal{H}) \cong \mathrm{Mon}(\mathcal{H}) \cong F_2/H$.

For a regular hypermap $\mathcal{H}$ we can identify the set Φ with $G := \mathrm{Aut}(\mathcal{H})$, and $\mathrm{Aut}(\mathcal{H})$ and $\mathrm{Mon}(\mathcal{H})$ with the left and right regular representations of

G, respectively. So there are automorphisms x and y in G, corresponding to ρ and λ, which generate the stabilizers of an incident pair of hypervertice and hyperedge respectively. In this case the triple (G, x, y) will be called an *algebraic hypermap*. A regular hypermap (G, x, y) is said to have *type* $(o(x), o(y), o(xy))$ and *genus* g where g is determined by the Euler-Poincaré formula:

$$2 - 2g = |G|(\frac{1}{o(x)} + \frac{1}{o(y)} + \frac{1}{o(xy)} - 1).$$

Let $\mathcal{H}_i = (G_i, x_i, y_i)$ $(i = 1, 2)$ be two regular hypermaps. Then $\mathcal{H}_1$ is a *covering* of $\mathcal{H}_2$ if the assignment $x_1 \mapsto x_2, y_1 \mapsto y_2$ extends to an epimorphism from G_1 onto G_2. The covering is *smooth* over hypervertices (resp. hyperedges, hyperfaces) if $o(x_1) = o(x_2)$ (resp. $o(y_1) = o(y_2)$, $o(x_1y_2) = o(x_2y_2)$); otherwise it is called branched over the hypervertices (resp. hyperedges, hyperfaces). Coverings which are smooth simultaneously over hypervertices, hyperedges and hyperfaces are called *smooth*.

An automorphism σ of F_2 sends H to H^σ, and hence transforms the hypermap $\mathcal{H}$ to a hypermap $\mathcal{H}^\sigma$ corresponding to H^σ. In particular if σ is an inner automorphism of F_2, then H^σ is conjugate to H, and hence $\mathcal{H}^\sigma$ is isomorphic to $\mathcal{H}$. It follows that the outer automorphism group $\Omega := \mathrm{Out}(F_2) = \mathrm{Aut}(F_2)/\mathrm{Inn}(F_2)$ acts as a *group of hypermap operations* on isomorphism classes of hypermaps. It has an faithful action on the abelianisation $F_2/F_2' \cong \mathbb{Z} \times \mathbb{Z}$, and hence $\Omega \cong GL(2, \mathbb{Z})$ [29].

It is well known that the automorphism group $\mathrm{Aut}(F_2)$ of F_2 is generated by the elementary Nielsen transformations of the form

$$\begin{aligned}
&\tau : X \to Y, Y \to X, \qquad \pi : X \to X, Y \to Y^{-1}, \quad \pi_1 : X \to X^{-1}, Y \to Y, \\
&\zeta : X \to XY, Y \to Y, \quad \eta : X \to X, Y \to YX.
\end{aligned}$$

Let

$$\varsigma = \zeta^{-1}\eta : X \to Y^{-1}, Y \to YX \quad \text{and} \quad \theta = \varsigma^2 : X \to Z^{Y^{-1}}, Y \to X^{Y^{-1}}.$$

The automorphism τ induces an operation ω_τ on hypermaps which transposes the hypervertices and the hyperedges while preserves the hyperfaces and orientation. The automorphism π (resp. π_1) induces an operation ω_π (resp. ω_{π_1}) which reverses the orientation around the hyperedgess (resp. hypervertices) but preserves it around the hypervertices (resp. hyperedges). These are sometimes called Petrie operations, because in the Walsh map of the hypermap they transpose the faces with the Petrie polygons (the closed zig-zag walks). Moreover, the automorphism $\iota = \pi\pi_1 : X \mapsto X^{-1}, Y \mapsto Y^{-1}$ induces an operation that reverses the orientation around the hypervertices and hyperedges, and thus transforms a hypermap to its mirror image. The

automorphism θ induces an operation ω_θ of order 3 which permutes the hypervertices, hyperfaces and hyperedges.

Define

$$\begin{array}{lll} \Lambda_1 = \langle \omega_\pi, \omega_\iota \rangle, & \Lambda_2 = \langle \omega_\iota, \omega_\tau \rangle, & \Lambda_3 = \langle \omega_\tau, \omega_\theta \rangle, \\ \Omega_1 = \langle \omega_\pi, \omega_\tau \rangle, & \Omega_2 = \langle \omega_\tau, \omega_\varsigma \rangle, & \mho = \langle \omega_\tau, \omega_\theta, \omega_\pi \rangle. \end{array}$$

Proposition 1. *In the subgroup lattice of Ω, the subgroups Ω_i $(i = 1, 2)$ and Λ_j $(j = 1, 2, 3)$ of Ω satisfy the Hasse graph depicted in Figure 1. In particular,*

(i) $\Lambda_1 \cong \Lambda_2 \cong \mathrm{V}_4$ *and* $\Lambda_3 \cong \mathrm{Sym}(3)$.

(ii) $\Omega_1 \cong \mathrm{D}_8$ *and* $\Omega_2 \cong \mathrm{D}_{12}$ *and they are maximal finite subgroups of* Ω.

(iii) $\Omega = \langle \omega_\varsigma, \omega_\tau, \omega_\pi \rangle$.

(iv) $[\Omega : \mho] = 2$.

Proof. The proof of (i) - (iii) can be found in [24]. To prove (iv), we take the images of X and Y in F_2/F_2' as a basis, and identify Ω with $GL(2, \mathbb{Z})$ so that the operations ω_τ, ω_π, ω_θ and ω_ς are represented by matrices of the form (see [24])

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} D &= S^2, \\ S^{-1}TS &= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = TD, \\ S^{-1}PS &= \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} = (TDT)^{-1}P(TDT). \end{aligned}$$

Since $\mho = \langle D, T, P \rangle$ and $\Omega = \langle S, T, P \rangle$, we have $\mho \trianglelefteq \Omega$ and $\Omega/\mho \cong \mathbb{Z}_2$. $\square$

If a regular hypermap is invariant under a hypermap operation, then we say that it possesses the corresponding external symmetry. More specifically, a regular hypermap $\mathcal{H}$ is *symmetric* if $\omega_\tau(\mathcal{H}) \cong \mathcal{H}$, *reflexible* if $\omega_\iota(\mathcal{H}) \cong \mathcal{H}$, *self-Petrie-dual* if $\omega_\pi(\mathcal{H}) \cong \mathcal{H}$, *triplly self-dual* if $\omega_\theta(\mathcal{H}) \cong \mathcal{H}$. Moreover, if a regular hypermap is invariant under all operations in Γ where $\Gamma \leq \Omega$, then it will be called Γ-*invariant*. Regular hypermaps which are invariant under all hypermap operations will be called *totally symmetric*.

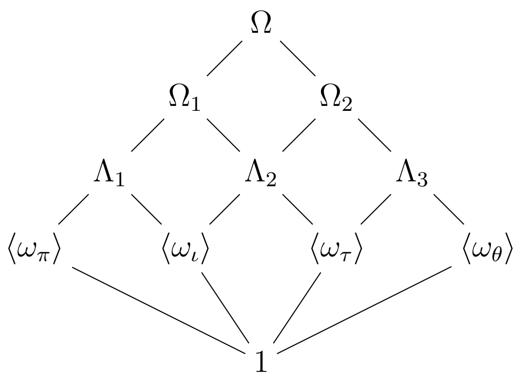

Figure 1: Finite subgroups of hypermap operations

Example 1. Let G be a metacyclic 2-group defined by the presentation

$$G = \langle x, y \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1}\rangle.$$

By Burnside's Basis Theorem $G = \langle y^i x^j, y^r x^s\rangle$ if and only if

$$is - jr \not\equiv 0 \pmod 2$$

So the number of generating pairs of G is equal to 3×2^5. Moreover, if $is - jr \not\equiv 0 \pmod 2$, then the assignment

$$x \mapsto x_1 = y^i x^j, y \mapsto y_1 = y^r x^s$$

extends to an automorphism of G if and only if i is even and both j, r are odd. Therefore, $|\mathrm{Aut}(G)| = 2^5$. Since $\mathrm{Aut}(G)$ acts semi-regularly on the generating pairs, up to isomorphism there are precisely three regular hypermaps $\mathcal{H}$ with $\mathrm{Aut}(\mathcal{H}) \cong G$. The representatives are $\mathcal{H}_1 = (G, x, y)$, $\mathcal{H}_2 = (G, y, x)$ and $\mathcal{H}_3 = (G, y, xy)$, all of type $(4, 4, 4)$, genus 3 and underlying bipartite graph $K_{4,4}$. Direct verification shows that ω_τ interchanges $\mathcal{H}_1$ and $\mathcal{H}_2$ while fixes $\mathcal{H}_3$, and both ω_π and ω_ι fix each $\mathcal{H}_i$ $(i = 1, 2, 3)$. Hence $\mathcal{H}_1$ and $\mathcal{H}_2$ are reflexible and self-Petrie-dual but not symmetric, while $\mathcal{H}_3$ is reflexible, symmetric and self-Petrie-dual, and hence Ω_1-invariant.

Example 2. The quaternion group Q_8, often defined by a presentation

$$\mathrm{Q}_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle,$$

underlies a unique regular hypermap $\mathcal{H} = (\mathrm{Q}_8, x, y)$ [17, Example 3]. It has type $(4, 4, 4)$ and genus 2. The underlying bipartite graph of $\mathcal{H}$ is isomorphic

to $K_{2,2}^{(2)}$, the complete bipartite graph $K_{2,2}$ with multiplicity two. Since the automorphism group of a regular hypermap is invariant under hypermap operations, the uniqueness implies that $\mathcal{H}$ is invariant under all hypermap operations, and hence totally symmetric. Using Tietze transformations [30, Chapter 1] the above presentation may be rewritten as the form

$$\mathrm{Q}_8 = \langle x, y \mid (xy)^4 = xy^{-1}xy = yx^{-1}yx = 1\rangle. \tag{1}$$

3 Preliminaries from group theory

In this section we summarize some prerequisites from group theory including terminology, notation and results to be used later. Throughout the paper groups considered are finite, except otherwise stated.

Let G be a group, and $x_i \in G$ $(i = 1, 2, \ldots, n)$. The *simple commutators* $[x_1, x_2, \ldots, x_{n-1}, x_n]$ $(n \geq 2)$ are defined inductively by the rule:

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 \quad \text{and} \quad [x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

Define $G_n = \langle [x_1, \ldots, x_n] \mid x_i \in G\rangle$. Note that $G_2 = G'$, the commutator subgroup of G. Then the series

$$G = G_1 \geq G_2 \geq G_3 \geq \cdots$$

is called the *lower central series* of G. A group G is called *nilpotent* if its lower central series terminates at the identity group, that is,

$$G = G_1 \geq G_2 \geq G_3 \geq \cdots \geq G_s \geq G_{s+1} = 1,$$

where the number s is called the *class* of G and is denoted by $c(G)$. It is well known that a finite group is nilpotent if and only if it is the direct product of its Sylow subgroups.

Lemma 2. [18, Chapter III, Lemma 1.11] *Let G be a group, $G = \langle M\rangle$ where M is a non-empty set of G. Then $G_n = \langle [x_1, x_2, \ldots, x_n]^g \mid x_i \in M, g \in G\rangle$. In particular, if $M = \{x_1, x_2\}$, then $G' = \langle [x_1, x_2]^g \mid g \in G\rangle$.*

Lemma 3. [14, Lemma 9] *Let $G = AB$ be a finite group where A and B are cyclic subgroups of G. If G is abelian, then G is cyclic if and only if $|A \cap B| = \gcd(|A|, |B|)$.*

A group G is called an *extension* a group A by B if $B \trianglelefteq G$ and $G/B \cong A$. The following result on cyclic extensions of groups is well known.

Lemma 4. [19, Theorem 3.36] *Let A and B be groups, where A is cyclic of order m, let $a \in B$ and $\sigma \in \mathrm{Aut}(B)$. Assume that*

$$a^\sigma = a \quad \textit{and} \quad x^{\sigma^m} = x^a, \quad \forall x \in B.$$

Then there exists an extension G of A by B, unique up to isomorphism, with the following properties: (i) $G/B = \langle gB \rangle \cong A$, (ii) $g^m = a$, (iii) $x^\sigma = x^g$.

As a particular case Hölder's theorem determines metacyclic groups, extensions of a cyclic group by a cyclic group.

Lemma 5 (Hölder's Theorem)**.** [36, Chapter III] *Let $m, n \geq 2$ be positive integers, $A \cong \mathbb{Z}_m$ and $B \cong \mathbb{Z}_n$ be cyclic groups. Then an extension G of A by B is determined by the solutions of the congruences*

$$r^m \equiv 1 \pmod{n} \quad \textit{and} \quad t(r-1) \equiv 0 \pmod{n}, \tag{2}$$

where $r, t \in \mathbb{Z}_n$. In particular, G has a presentation

$$G = \langle u, v \mid u^n = 1, v^m = u^t, v^{-1}uv = u^r \rangle. \tag{3}$$

Conversely, each extension G of A by B is defined by the presentation (3) with the parameters satisfying (2).

4 Normalized multicyclic coverings

In this section we employ the theory on factor groups in combinatorial group theory to study regular coverings between regular hypermaps.

Let $\mathcal{H}_i = (G_i, x_i, y_i)$ $(i = 1, 2)$ be two regular hypermaps, where G_1 is defined by a presentation

$$G_1 = \langle x_1, y_1 \mid R_1(x_1, y_1), \ldots, R_r(x_1, y_1) \rangle,$$

where $R_i(x_1, y_1)$ $(i = 1, \ldots, r)$ are *relators* in x_1 and y_1, that is, $R_i(x_1, y_1) = 1$. If $\mathcal{H}_1 \to \mathcal{H}_2$ is a regular covering, then there is an epimorphism $G_1 \to G_2$ sending x_1 to x_2 and y_1 to y_2. So G_2, being a factor group of G_1, has a presentation

$$G_2 = \langle x_2, y_2 \mid R_1(x_2, y_2), \ldots, R_m(x_2, y_2), S_1(x_2, y_2), \ldots, S_s(x_2, y_2) \rangle,$$

where $R_1(x_2, y_2), \ldots, R_r(x_2, y_2)$ are the same relators as before (but in x_2 and y_2), and $S_1(x_2, y_2), \ldots, S_s(x_2, y_2)$ are additional relators. It follows that the covering transformation group K, being the kernel of the epimorphism, is

the normal closure $\{S_1(x_1,y_1),\dots,S_s(x_1,y_1)\}^{G_1}$, the normal subgroup of G_1 generated by the words $S_1(x_1,y_1),\dots,S_s(x_1,y_1)$ (cf. [30, Chapter 2]). Define

$$u_i = S_i(x_1,y_1) \quad \text{and} \quad U_i = \langle u_i\rangle,\ i=1,\dots,s.$$

Clearly U_i $(i=1,\dots,s)$ are cyclic subgroups of G_1 contained in K.

Proposition 6. *With the above notation, if K is a cyclic group, then*

$$U_i \trianglelefteq G_1\ (i=1,\dots,s) \quad \textit{and} \quad K = U_1U_2\cdots U_s. \tag{4}$$

Proof. Note that the subgroups U_i $(i=1,2,\dots,s)$, being subgroups of a cyclic group K, is characteristic in K. Since $K \trianglelefteq G_1$, we have $U_i \trianglelefteq G_1$, and hence $U_1U_2\cdots U_s \trianglelefteq G_1$. Since K is the minimal normal subgroup of G_1 containing U_i $(i=1,\dots,s)$, we have $K = U_1U_2\cdots U_s$, as required. □

The converse of Proposition 3 is not necessarily true.

Example 3. Let $\mathcal{H}=(G,x,y)$ where G is a group with a presentation

$$\langle x,y \mid x^4=y^4=(xy)^4=[x,y^2]=[y,x^2]=1\rangle.$$

This is a regular hypermap of type $(4,4,4)$ and genus 5, appearing as entry RPH5.8 in Conder's census of orientable proper regular hypermaps [3]. Let $K=\{xy^{-1}xy, yx^{-1}yx\}^G$, then $G/K\cong \mathrm{Q}_8$, so $\mathcal{H}$ is a smooth covering of the quaternion hypermap. Let $u=xy^{-1}xy$ and $v=yx^{-1}yx$, then

$$\begin{aligned} u^x &= y^{-1}xyx = y^{-1}x^{-1}x^2yx = y^{-1}x^{-1}yx^{-1} = u^{-1},\\ u^y &= y^{-1}xy^{-1}xy^2 = y^{-1}x^{-1}x^2yx = y^{-1}x^{-1}yx^{-1} = u^{-1}. \end{aligned}$$

Hence $\langle u\rangle \trianglelefteq G$. Similarly $v^x=v^{-1}$ and $v^y=v^{-1}$, so $\langle v\rangle \trianglelefteq G$. Moreover, $u^2=(xy^{-1}xy)^2=(xyy^{-2}xy)^2=(xy)^4y^{-4}=1$, and similarly $v^2=1$. It is clear that $\langle u\rangle\cap\langle v\rangle=1$. Therefore $K=\langle u\rangle\langle v\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$, a non-cyclic group satisfying the condition (4).

Definition 1. With the above notation, if the covering transformation group K of a regular covering $\mathcal{H}_1\to\mathcal{H}_2$ between two regular hypermaps is defined by (4), then the covering will be called an $\{S_1(x_1,y_1),\dots,S_s(x_1,y_1)\}$-*normalized s-cyclic covering*, or simply *normalized multicyclic covering* if we do not wish to specify the relators.

In what follows we study the structure of the covering transformation group of a normalized multicyclic covering.

Theorem 7. *The covering transformation group of a normalized s-cyclic covering between regular hypermaps is a nilpotent group of class at most s.*

Proof. With the above notation, by the definition we have $U_i \trianglelefteq G_1$ $(i = 1, 2, \ldots, s)$ and $K = U_1U_2 \cdots U_s$. Since $K \leq G_1$, we have $U_i \trianglelefteq K$. Let p be an arbitrary prime factor of $|K|$, and P_i be the Sylow-p subgroups of U_i $(i = 1, 2, \ldots, s)$. Since U_i are cyclic, P_i char U_i, and hence $P_i \trianglelefteq K$. It follows that $P = P_1P_2 \cdots P_s \trianglelefteq K$. Clearly, P is the Sylow p-subgroup of K. Therefore K is nilpotent.

Now we show that $c(K) \leq s$. Since $\langle u_i \rangle \trianglelefteq K$, for each integer pair (i, j) $(1 \leq i, j \leq s)$, there are integers $\lambda_{i,j}$ such that $u_i^{u_j} = u_i^{\lambda_{ij}}$. It follows that

$$u_i^{\lambda_{ij}-1} = [u_i, u_j] = u_j^{1-\lambda_{ji}} \quad \text{and} \quad [u_i^k, u_j] = u_i^{k(\lambda_{ij}-1)},$$

where k is an integer. Thus, for any permutation $\sigma \in \mathrm{Sym}(s)$ and any integer $i \in I = \{1, 2, \ldots, s\}$,

$$\begin{aligned}
&[u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(i)}, \ldots, u_{\sigma(s)}] \\
=&[u_{\sigma(1)}^{\lambda_{\sigma(1),\sigma(2)}-1}, u_{\sigma(3)}, \ldots, u_{\sigma(i)}, \ldots, u_{\sigma(s)}] \\
=&[u_{\sigma(1)}^{\left(\lambda_{\sigma(1),\sigma(2)}-1\right)\left(\lambda_{\sigma(1),\sigma(3)}-1\right)}, u_{\sigma(4)}, \ldots, u_{\sigma(i)}, \ldots, u_{\sigma(s)}] \\
=&[u_{\sigma(1)}^{\prod_{k=2}^{i-1}\left(\lambda_{\sigma(1),\sigma(k)}-1\right)}, u_{\sigma(i)}, \ldots, u_{\sigma(s)}] \\
=&[u_{\sigma(i)}^{\left(1-\lambda_{\sigma(i),\sigma(1)}\right)\prod_{k=2}^{i-1}\left(\lambda_{\sigma(1),\sigma(k)}-1\right)}, u_{\sigma(i+1)}, \ldots, u_{\sigma(s)}] \\
=&u_{\sigma(i)}^{\left(1-\lambda_{\sigma(i),\sigma(1)}\right)\prod_{k=2}^{i-1}\left(\lambda_{\sigma(1),\sigma(k)}-1\right)\prod_{l=i+1}^{s}\left(\lambda_{\sigma(i),\sigma(l)}-1\right)}.
\end{aligned}$$

Let $e(\sigma, i) = \left(1 - \lambda_{\sigma(i),\sigma(1)}\right) \prod_{k=2}^{i-1} \left(\lambda_{\sigma(1),\sigma(k)} - 1\right) \prod_{l=i+1}^{s} \left(\lambda_{\sigma(i),\sigma(l)} - 1\right)$. Since σ is a permutation of the numbers in I, for any $j \in I$, there is a number $i \in I$ such that $\sigma(i) = j$, and hence

$$[u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(s)}, u_j] = [u_{\sigma(i)}^{e(\sigma,i)}, u_j] = [u_j^{e(\sigma,i)}, u_j] = 1.$$

Therefore, by Lemma 2, we have $K_{s+1} = 1$. $\square$

In what follows we classify the covering transformation group of a normalized bicyclic covering between regular hypermaps. We see in Proposition 6 and Example 3 that the covering transformation group of a normalized bicyclic covering can be a (cyclic or non-cyclic) abelian group. The following example shows that a normalized bicyclic covering can have a non-abelian nilpotent covering transformation group.

Example 4. [15, Example 1] Let $\mathcal{M} = (G, x, y)$ be a regular map where G is defined by the presentation

$$G = \langle x, y \mid x^{12} = y^2 = z^{12} = 1, (x^3)^z = x^9, (z^3)^x = z^9, z^6 = x^6, z = (xy)^{-1} \rangle.$$

In Coxeter-Moser's notation this is a reflexible regular map of type $\{12, 12\}$ and genus 17, appearing as entry R17.35 in Conder's list of regular maps [3]. Let $K = \{u, v\}^G$ where $u = x^3$ and $v = z^3$. Then $G/K \cong \mathrm{Alt}_4$, so it is a regular covering of the tetrahedral map, branched simultaneously over vertices and face-centres. By the presentation it is clear that $\langle u \rangle, \langle v \rangle \trianglelefteq G$, and hence $K = \langle u \rangle \langle v \rangle$. So it is a normalized bicyclic covering of the tetrahedral map (this was called an almost totally branched covering in [14, 15]). Since $u^v = (x^3)^{z^3} = x^{81} = x^{-3} = u^{-1}$ and $v^u = (z^3)^{x^3} = z^{81} = z^{-3} = v^{-1}$, we have

$$K = \langle u, v \mid u^4 = v^4 = uv^{-1}uv = vu^{-1}vu = 1 \rangle \cong \mathrm{Q}_8,$$

a non-abelian group satisfying the condition (4).

The following theorem classifies the covering transformation group K of a normalized bicyclic covering between regular hypermaps. By Theorem 7 it suffices to consider the case when K is a p-group.

Theorem 8. *Let p be a prime. If the covering transformation group of a normalized bicyclic covering is p-group, then it is isomorphic to a group with a presentation*

$$\langle u, v \mid u^{p^{a+c}} = v^{p^{b+c}} = 1, u^{p^a} = v^{p^b}, u^v = u^{1+p^{a+d}}, v^u = v^{1-p^{b+d}} \rangle, \qquad (5)$$

where a, b, c, d are nonnegative integers such that

$$0 \leq d \leq c \leq a + d \leq b + d. \qquad (6)$$

Moreover, two such groups are isomorphic if and only if the corresponding numerical parameters p, a, b, c, d are identical.

Proof. Assume that K is the covering transformation p-group of a normalized bicyclic covering $\mathcal{H}_1 \to \mathcal{H}_2$ between regular hypermaps where $\mathcal{H}_1 = (G_1, x_1, y_1)$. By definition $K = \langle u_1, v_1 \rangle$ where $\langle u_1 \rangle \trianglelefteq G_1$ and $\langle v_1 \rangle \trianglelefteq G_1$. Since $K \leq G_1$, we have $\langle u_1 \rangle \trianglelefteq K$ and $\langle v_1 \rangle \trianglelefteq K$. Without loss of generality we may assume that $o(u_1) \leq o(v_1)$. By Hölder's Theorem, the p-group K, being a metacyclic group, has a presentation

$$K = \langle u_1, v_1 \mid u_1^{p^{a+c}} = v_1^{p^{b+c}} = 1, u_1^{p^a} = v_1^{\epsilon p^b}, v_1^{-1}u_1v_1 = u_1^{\lambda_1}, u_1^{-1}v_1u_1 = v_1^{\gamma_1} \rangle,$$

where $0 \leq c$, $0 \leq a \leq b$, and the numbers ϵ, λ_1, γ_1 are coprime to p. Let $u_2 = u_1$ and $v_2 = v_1^\epsilon$, then the presentation is transformed into the form

$$K = \langle u_2, v_2 \mid u_2^{p^{a+c}} = v_2^{p^{b+c}} = 1, u_2^{p^a} = v_2^{p^b}, v_2^{-1}u_2v_2 = u_2^\lambda, u_2^{-1}v_2u_2 = v_2^\gamma\rangle, \quad (7)$$

where $\lambda = \lambda_1^\epsilon$ and $\gamma = \epsilon\gamma_1$. By the presentation we have $u_2 = v_2^{-p^b}u_2v_2^{p^b} = u_2^{\lambda^{p^b}}$ and $v_2 = u_2^{-p^a}v_2u_2^{p^a} = v_2^{\gamma^{p^a}}$. Hence

$$\lambda^{p^b} \equiv 1 \pmod{p^{a+c}} \quad \text{and} \quad \gamma^{p^a} \equiv 1 \pmod{p^{b+c}}. \quad (8)$$

Since $u_2^{p^a} = v_2^{-1}u_2^{p^a}v_2 = u_2^{\lambda p^a}$ and $v_2^{p^b} = u_2^{-1}v_2^{p^b}u_2 = v_1^{\gamma p^b}$, we obtain that

$$\lambda \equiv 1 \pmod{p^c} \quad \text{and} \quad \gamma \equiv 1 \pmod{p^c}. \quad (9)$$

Recall that $\langle u_2\rangle \cap \langle v_2\rangle = \langle u_2^{p^a}\rangle = \langle v_2^{p^b}\rangle$. Since $u_2^{\lambda-1} = [u_2, v_2] = v_2^{1-\gamma}$, we have

$$\lambda \equiv 1 \pmod{p^a} \quad \text{and} \quad \gamma \equiv 1 \pmod{p^b}. \quad (10)$$

It follows that $v_2^{1-\gamma} = u_2^{\lambda-1} = (u_2^{p^a})^{\frac{\lambda-1}{p^a}} = (v_2^{p^b})^{\frac{\lambda-1}{p^a}}$, and hence

$$\frac{\lambda-1}{p^a} + \frac{\gamma-1}{p^b} \equiv 0 \pmod{p^c}. \quad (11)$$

By (10) we may assume that $\lambda = ip^{a+d}+1$ and $\gamma = jp^{b+e}$ where $0 \leq d, e \leq c$, $i \in \mathbb{Z}_{p^{c-d}}^*$ and $j \in \mathbb{Z}_{p^{c-e}}^*$. Upon substitution the congruence (11) reduces to

$$ip^d + jp^e \equiv 0 \pmod{p^c}.$$

Since $0 \leq d, e \leq c$ and $\gcd(i, p) = \gcd(j, p) = 1$, we have $d = e$, and hence $j \equiv -i \pmod{p^{c-d}}$. Using substitution in (9) we obtain $ip^{a+d} \equiv 0 \pmod{p^c}$, and hence $c \leq a+d$. Summarizing the above numerical conditions we obtain (6). In particular, since

$$\lambda^{p^b} - 1 = (1 + ip^{a+d})^{p^b} - 1 = \sum_{k=1}^{p^b} \binom{p^b}{k} (ip^{a+d})^k \equiv 0 \pmod{p^{a+c}}$$

and

$$\gamma^{p^a} - 1 = (1 - ip^{b+d})^{p^a} - 1 = \sum_{k=1}^{p^a} \binom{p^a}{k} (-ip^{b+d})^k \equiv 0 \pmod{p^{b+c}},$$

the conditions in (8) are fulfilled. Let $u = u_2^{i_1}$ and $v = v_2^{i_1}$ where $i_1 i \equiv 1 \pmod{p^{c-d}}$. Then the presentation (7) is transformed into (5).

Finally, it is clear that $K' = \langle u^{p^{a+d}}\rangle \cong \mathbb{Z}_{p^{c-d}}$, and $K/K' \cong \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^{b+d}}$, so the numbers $a+d$, $c-d$, a and $b+d$ are group invariants, and hence a, b, c, d are group invariants as well, as claimed. $\square$

5 Classification

In this section we present a classification of a particular family of normalized bicyclic coverings of the quaternion hypermap with an abelian covering transformation group.

Theorem 9. *The isomorphism classes of abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic coverings $\mathcal{H}$ of the quaternion hypermap are in one-to-one correspondence with the octuples $(m, n, d, \alpha, \beta, \gamma, \delta, \epsilon)$, where m, n and d are positive integers, $\alpha, \beta \in \mathbb{Z}^*_{md}$, $\gamma, \delta \in \mathbb{Z}^*_{nd}$ and $\epsilon \in \mathbb{Z}^*_d$, satisfying the following numerical conditions:*

$$\alpha^2 \equiv 1 \pmod{md}, \qquad \delta^2 \equiv 1 \pmod{nd}, \tag{12}$$
$$\beta^2 \equiv 1 \pmod{md}, \qquad \gamma^2 \equiv 1 \pmod{nd}, \tag{13}$$
$$\beta \equiv 1 \pmod{m}, \qquad \gamma \equiv 1 \pmod{n}, \tag{14}$$
$$\alpha \equiv \gamma \pmod{d}, \qquad \beta \equiv \delta \pmod{d}, \tag{15}$$

and

$$\frac{\beta - 1}{m}\epsilon + \frac{\gamma - 1}{n} \equiv 0 \pmod{d}. \tag{16}$$

Moreover, the automorphism group $\mathrm{Aut}(\mathcal{H})$ *has a presentation*

$$\begin{aligned}&\langle x, y \mid x^4 = u^{\alpha+1}v^{\gamma-\delta}, y^4 = u^{\beta-\alpha}v^{\delta+1}, (xy)^4 = u^{\alpha+1}v^{\gamma+\delta}, u^{md} = v^{nd} = 1,\\ &u^x = u^\alpha, u^y = u^\beta, v^x = v^\gamma, v^y = v^\delta, u^m = v^{n\epsilon}, u = xy^{-1}xy, v = yx^{-1}yx\rangle,\end{aligned} \tag{17}$$

with $K = \langle u, v\rangle \trianglelefteq \mathrm{Aut}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})/K \cong \mathrm{Q}_8$. In particular, the covering is cyclic if and only if $\gcd(m, n) = 1$.

Proof. Let $\mathcal{H} = (G, x, y)$ and define

$$u = xy^{-1}xy \quad \text{and} \quad v = yx^{-1}yx. \tag{18}$$

Since the covering is an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering, by Definition 1 we have $\langle u\rangle, \langle v\rangle \trianglelefteq G$ and the covering transformation group $K = \langle u, v\rangle$ is abelian. Assume that $o(u) = md$ and $o(v) = nd$ where $d = |\langle u\rangle \cap \langle v\rangle|$. Then there are integers $\alpha, \beta \in \mathbb{Z}^*_{md}$, $\gamma, \delta \in \mathbb{Z}^*_{nd}$ and $\epsilon \in \mathbb{Z}^*_d$ such that

$$u^x = u^\alpha, \quad u^y = u^\beta, \quad v^x = v^\gamma, \quad v^y = v^\delta \quad \text{and} \quad u^m = v^{n\epsilon}.$$

We deduce from (18) that $x^{-2}u = [x, y] = v^{-1}y^2$. This is equivalent to

$$y^2 = vx^{-2}u \quad \text{or} \quad x^2 = uy^{-2}v. \tag{19}$$

Also note that the relations in (18) can be rewritten as the form

$$x^y = x^{-1}u \quad \text{and} \quad y^x = y^{-1}v. \tag{20}$$

So we have

$$\begin{aligned}
(y^2)^x &= (y^x)^2 = (y^{-1}v)^2 = (y^{-1}vy)y^{-2}v = v^{\delta}(u^{-1}x^2v^{-1})v = x^2u^{-1}v^{\delta},\\
(y^2)^x &= (vx^{-2}u)^x = v^x x^{-2}u^x = x^{-2}u^{\alpha}v^{\gamma}.
\end{aligned}$$

By equating the right-hand sides we obtain

$$x^4 = u^{\alpha+1}v^{\gamma-\delta}. \tag{21}$$

Similarly,

$$\begin{aligned}
(x^2)^y &= (x^y)^2 = (x^{-1}u)^2 = (x^{-1}ux)x^{-2}u = u^{\alpha}(v^{-1}y^2u^{-1})u = y^2u^{\alpha}v^{-1},\\
(x^2)^y &= (uy^{-2}v)^y = u^y y^{-2}v^y = y^{-2}u^{\beta}v^{\delta}.
\end{aligned}$$

Equating the right-hand sides yields

$$y^4 = u^{\beta-\alpha}v^{\delta+1}. \tag{22}$$

Moreover, by taking conjugation on the relations $v^x = v^{\gamma}$ and $u^y = u^{\beta}$ we have $v^{x^4} = v^{\gamma^4}$ and $u^{y^4} = u^{\beta^4}$. Since $K = \langle u, v\rangle$ is an abelian group, by (21) and (22) we have $v = v^{x^4}$ and $u = u^{y^4}$. Therefore $u = u^{\beta^4}$ and $v = v^{\gamma^4}$, and hence $\beta^4 \equiv 1 \pmod{md}$ and $\gamma^4 \equiv 1 \pmod{nd}$. It follows that

$$\begin{aligned}
u &= u^v = u^{yx^{-1}yx} = (u^{\beta})^{x^{-1}yx} = (u^{\beta\alpha^3})^{yx} = u^{\alpha^4\beta^2} = u^{\beta^2},\\
v &= v^u = v^{xy^{-1}xy} = (v^{\gamma})^{y^{-1}xy} = (v^{\gamma\delta^3})^{xy} = v^{\gamma^2\delta^4} = v^{\gamma^2}.
\end{aligned}$$

Consequently $\beta^2 \equiv 1 \pmod{md}$ and $\gamma^2 \equiv 1 \pmod{nd}$. From these and (19) we deduce that

$$\begin{aligned}
u^{\alpha^2} &= u^{x^2} = u^{uy^{-2}v} = u^{\beta^2} = u,\\
v^{\delta^2} &= v^{y^2} = v^{vx^{-2}u} = v^{\gamma^2} = v.
\end{aligned}$$

Hence $\alpha^2 \equiv 1 \pmod{md}$ and $\delta^2 \equiv 1 \pmod{nd}$.

Further, we deduce from the relation $u^m = v^{n\epsilon}$ that

$$\begin{aligned}
u^{m\alpha} &= (u^x)^m = (u^m)^x = (v^{n\epsilon})^x = (v^x)^{n\epsilon} = v^{n\epsilon\gamma} = u^{m\gamma},\\
u^{m\beta} &= (u^y)^m = (u^m)^y = (v^{n\epsilon})^y = (v^y)^{n\epsilon} = v^{n\epsilon\delta} = u^{m\delta}.
\end{aligned}$$

Hence $\alpha \equiv \gamma \pmod{d}$ and $\beta \equiv \delta \pmod{d}$. From (19) with (20) we deduce that

$$\begin{aligned} x^{y^2} &= (x^{-1}u)^y = (x^y)^{-1}u^y = u^{-1}xu^{\beta} = xu^{\beta-\alpha}, \\ x^{y^2} &= x^{vx^{-2}u} = u^{-1}x^2v^{-1}xvx^{-2}u = u^{-1}v^{-1}xvu = xu^{1-\alpha}v^{1-\gamma}. \end{aligned}$$

By equating the right-hand sides we then obtain

$$u^{\beta-1} = v^{1-\gamma}. \tag{23}$$

Since $\langle u\rangle \cap \langle v\rangle = \langle u^m\rangle = \langle v^n\rangle$, the above relation implies that $\beta \equiv 1 \pmod{m}$ and $\gamma \equiv 1 \pmod{n}$. So using substitution $u^m = v^{n\epsilon}$ we obtain

$$v^{1-\gamma} = u^{\beta-1} = (u^m)^{\frac{\beta-1}{m}} = v^{\frac{(\beta-1)n}{m}\epsilon},$$

and consequently

$$\frac{\beta-1}{m}\epsilon + \frac{\gamma-1}{n} \equiv 0 \pmod{d}.$$

Finally by (20) and (21) we have

$$(xy)^4 = (xyxy)^2 \overset{(20)}{=} (x^2y^{-1}vy)^2 = (x^2v^{\delta})^2 = x^4v^{2\delta} \overset{(21)}{=} u^{\alpha+1}v^{\gamma+\delta}. \tag{24}$$

Therefore G has the presentation (17).

Conversely, by Lemma 4 it is straightforward to verify that the group with the presentation (17) is a well-defined extension of Q_8 by an abelian group K of order mnd, provided that the numerical conditions (12) through (16) are fulfilled. It is easily seen from the presentation that the corresponding hypermap $\mathcal{H}$ is an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering of the quaternion hypermap.

To prove the one-to-one correspondence, we let

$$\mathcal{H}_i = \mathcal{H}(m_i, n_i, d_i; \alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i) = (G_i, x_i, y_i)$$

be coverings specified by the parameters $m_i, n_i, d_i, \alpha_i, \beta_i, \gamma_i, \delta_i$ and ϵ_i ($i = 1, 2$). If $\mathcal{H}_1 \cong \mathcal{H}_2$, then the assignment $x_1 \mapsto x_2, y_1 \mapsto y_2$ extends to a group isomorphism $\phi : G_1 \to G_2$. In particular $\phi(u_1) = \phi(u_2)$ and $\phi(v_1) = \phi(v_2)$ where $u_i = x_iy_i^{-1}x_iy_i$ and $v_i = y_ix_i^{-1}y_ix_i$ ($i = 1, 2$). We have

$$d_1 = |\langle u_1\rangle \cap \langle v_1\rangle| = |\langle \phi(u_1)\rangle \cap \langle \phi(v_1)\rangle| = |\langle u_2\rangle \cap \langle v_2\rangle| = d_2$$

and

$$\begin{aligned} m_1d_1 &= o(u_1) = o(\phi(u_1)) = o(u_2) = m_2d_2, \\ n_1d_1 &= o(v_1) = o(\phi(v_1)) = o(v_2) = n_2d_2. \end{aligned}$$

Hence $d_1 = d_2$, $m_1 = m_2$ and $n_1 = n_2$. To simplify the notation, let $d = d_1$, $m = m_1$ and $n = n_1$. The relation $u_1^{x_1} = u_1^{\alpha_1}$ in G_1 is mapped by ϕ to a relation $\phi(u_1^{x_1}) = \phi(u_1^{\alpha_1})$ in G_2, so $u_2^{x_2} = u_2^{\alpha_1}$. Since $u_2^{x_2} = u_2^{\alpha_2}$, we have $\alpha_1 \equiv \alpha_2 \pmod{md}$. Using similar arguments it is easy to verify that $\beta_1 \equiv \beta_2 \pmod{md}$, $\gamma_1 \equiv \gamma_2 \pmod{nd}$, $\delta_1 \equiv \delta_2 \pmod{nd}$ and $\epsilon_1 \equiv \epsilon_2 \pmod{d}$.

Conversely, if the corresponding parameters are identical, then clearly the assignment $x_1 \mapsto x_1, y_1 \mapsto y_2$ extends to an isomorphism from G_1 onto G_2, and hence $\mathcal{H}_1 \cong \mathcal{H}_2$.

Finally, by Lemma 3, the covering transformation group K is cyclic if and only if $\gcd(m, n) = 1$, as claimed. $\square$

In what follows we denote by $\mathcal{H}(m, n, d; \alpha, \beta, \gamma, \delta, \epsilon)$ the regular coverings classified in Theorem 9 which are specified by the parameters. The following corollary follows from Theorem 9 and the Euler-Poincaré Formula.

Corollary 10. *The hypermaps $\mathcal{H}(m, n, d; \alpha, \beta, \gamma, \delta, \epsilon)$ determined in Theorem 9 have type $(4p, 4q, 4r)$ and genus*

$$g = mnd\Big(4 - (\frac{1}{p} + \frac{1}{q} + \frac{1}{r})\Big) + 1,$$

where

$$p = o(u^{\alpha+1}v^{\gamma-\delta}), \quad q = o(u^{\beta-\alpha}v^{\delta+1}) \quad \textit{and} \quad r = o(u^{\alpha+1}v^{\gamma+\delta}).$$

Example 5. Consider the case $d = 1$ and $\alpha = \beta = \gamma = \delta = \epsilon = 1$. One can easily verify that the conditions (12) - (16) are all fulfilled. Upon substitution we have

$$x^4 = u^2, \quad y^4 = v^2 \quad \text{and} \quad (xy)^4 = u^2v^2.$$

Since $d = 1$, $\langle u\rangle \cap \langle v\rangle = 1$, and hence $o(u^2v^2) = [o(u^2), o(v^2)]$, the least common multiple of $o(u^2)$ and $o(v^2)$. Note that $o(u) = m$ and $o(v) = n$. By distinguishing the parity of m and n, we determine the type and genus of the associated regular hypermaps, as summarized in Table 1.

Table 1: Type and genus

Class	m	n	Type	Genus
i	odd	odd	$(4m, 4n, 4[m, n])$	$4mn - m - n - (m, n) + 1$
ii	odd	even	$(4m, 2n, 4[m, \frac{n}{2}])$	$4mn-2m-n-2(m, \frac{n}{2})+1$
iii	even	odd	$(2m, 4n, 4[\frac{m}{2}, n])$	$4mn-m-2n-2(\frac{m}{2}, n)+1$
iv	evn	even	$(2m, 2n, 4[\frac{m}{2}, \frac{n}{2}])$	$4mn-2m-2n-4(\frac{m}{2}, \frac{n}{2})+1$

6 Coverings with exteral symmetries

In this section we determine the coverings of the quaternion hypermap classified in Theorem 9 which possess certain external symmetries.

Theorem 11. *The hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ determined in Theorem 9 is reflexible if and only if $\beta \equiv \gamma \pmod{d}$.*

Proof. If the hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon) = (G,x,y)$ is reflexible, then the assignment $\iota : x \mapsto x^{-1}, y \mapsto y^{-1}$ extends to an automorphism of G. In particular, we have $\iota(u) = u_1$ and $\iota(v) = v_1)$ where

$$u_1 = x^{-1}yx^{-1}y^{-1} \quad \text{and} \quad v_1 = y^{-1}xy^{-1}x^{-1}.$$

Note that the relation $u^m = v^{n\epsilon}$ is mapped by ι to $u_1^m = v_1^{n\epsilon}$. By (18) we have

$$\begin{aligned} u_1 &= x^{-1}(xyu^{-1})y^{-1} = yu^{-1}y^{-1} = u^{-\beta}, \\ v_1 &= y^{-1}(yxv^{-1})x^{-1} = xv^{-1}x^{-1} = v^{-\gamma}. \end{aligned}$$

Hence upon substitution we have

$$u^{-m\beta} = u_1^m = v_1^{n\epsilon} = v^{-n\epsilon\gamma} = u^{-m\gamma},$$

and consequently $\beta \equiv \gamma \pmod{d}$.

Conversely, it can be easily verified that if $\beta \equiv \gamma \pmod{d}$, then the above assignment extends to an automorphism of G, and hence the regular hypermap is reflexible, as required. $\square$

Theorem 12. *The hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ determined in Theorem 9 is symmetric if and only if $m = n$, $\alpha \equiv \delta \pmod{md}$, $\beta \equiv \gamma \pmod{md}$ and $\epsilon^2 \equiv 1 \pmod{d}$.*

Proof. If the hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon) = (G,x,y)$ is symmetric, then the assignment $\tau : x \mapsto y, y \mapsto x$ extends to an automorphism of G. In particular, we have $\tau(u) = v$ and $\tau(v) = u$, which implies that $m = n$.

Note that the relations $u^x = u^\alpha$ and $u^y = u^\beta$ are mapped by τ to relations of the form $v^y = v^\alpha$ and $v^x = v^\beta$. Since $v^y = v^\delta$ and $v^x = v^\gamma$, we obtain that $\alpha \equiv \delta \pmod{md}$ and $\beta \equiv \gamma \pmod{md}$.

Finally, the relation $u^m = v^{m\epsilon}$ is mapped to $v^m = u^{m\epsilon}$, so we have $u^m = v^{m\epsilon} = u^{m\epsilon^2}$, and hence $\epsilon^2 \equiv 1 \pmod{d}$.

Conversely, if the stated equalities are satisfied, then it is straightforward to verify that the above assignment extends to an automorphism of G, and hence the hypermap is symmetric, as required. $\square$

Theorem 13. *The hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ determined in Theorem 9 is self-Petrie-dual if and only if $\alpha \equiv -1 \pmod{d}$.*

Proof. If $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon) = (G,x,y)$ is self-Petrie-dual, then the assignment $\pi : x \mapsto x, y \mapsto y^{-1}$ extends to an automorphism of G. In particular, we have $\pi(u) = u_1$ and $\pi(v) = v_1$ where

$$u_1 = xyxy^{-1} \quad \text{and} \quad v_1 = y^{-1}x^{-1}y^{-1}x.$$

The relation $u^m = v^{n\epsilon}$ is mapped by π to $u_1^m = v_1^{n\epsilon}$. By (18) we have

$$\begin{aligned} u_1 &= y(x^{-1}ux)y^{-1} = u^{\alpha\beta}, \\ v_1 &= y^{-1}(v^{-1}yx^{-1})x = v^{-\delta}. \end{aligned}$$

Upon substitution the relation $u_1^m = v_1^{n\epsilon}$ is transformed into the form $u^{m\alpha\beta} = v^{-n\epsilon\delta} = u^{-m\delta}$, and hence $\alpha\beta + \delta \equiv 0 \pmod{d}$. Combining this with the congruence $\beta \equiv \delta \pmod{d}$ in (15) we have $\beta(\alpha+1) \equiv 0 \pmod{d}$. Since $\beta \in \mathbb{Z}_{md}^*$, we obtain that $\alpha \equiv -1 \pmod{d}$.

Conversely, it is straightforward to verify that if $\alpha \equiv -1 \pmod{d}$, then the above assignment extends to an automorphism of G, and hence the regular hypermap is self-Petrie-dual. □

In what follows we determine those coverings which are triplly self-dual. We first prove the following lemma.

Lemma 14. *Let $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ be the regular hypermaps determined in Theorem 9, then the following congruences on the parameters are derivable from* (12) *through* (16)*:*

$$(\alpha+1)(\beta-1) \equiv 0 \pmod{md}, \tag{25}$$

$$(\delta+1)(\gamma-1) \equiv 0 \pmod{nd}. \tag{26}$$

Proof. By (14) we have $\gamma \equiv 1 \pmod{n}$, so $(\gamma-1)/n$ is an integer. From $\gamma^2 \equiv 1 \pmod{nd}$ we deduce that

$$(\gamma+1)\frac{\gamma-1}{n} \equiv 0 \pmod{d}.$$

By Eqs. (15) and (16) we have $\gamma \equiv \alpha \pmod{d}$ and $\frac{\gamma-1}{n} \equiv -\frac{\beta-1}{m}\epsilon \pmod{d}$. Using substitution the above congruence is transformed into the form

$$(\alpha+1)\frac{\beta-1}{m}\epsilon \equiv 0 \pmod{d}.$$

Since $\epsilon \in \mathbb{Z}_d^*$, this is equivalent to (25). Using similar arguments one may derive (26) from the congruence $\beta^2 \equiv 1 \pmod{md}$, and we leave the details to the reader. □

Theorem 15. *The hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ determined in Theorem 9 is triplly self-dual if and only if the parameters fulfil the following additional conditions:*

$$\begin{aligned}
&m = n,\\
&\gamma = \alpha+\beta-1 \pmod{md},\\
&\delta \equiv \alpha \pmod{md},\\
&\alpha \equiv \beta \pmod{m},\\
&(\alpha-\beta)\epsilon \equiv \alpha - 1 \pmod{md},\\
&\epsilon^2 + (2-\alpha)\epsilon + \alpha \equiv 0 \pmod{d}.
\end{aligned}$$

Proof. If the regular hypermap $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon) = (G,x,y)$ is triplly self-dual, then the assignment $\theta : x \mapsto (xy)^{-1}, y \mapsto x$ extends to an automorphism of G. In particular, $\theta(u) = u_1$ and $\theta(v) = v_1$ where

$$u_1 = y^{-1}x^{-2}y^{-1} \quad \text{and} \quad v_1 = x^2yxy^{-1}x^{-1}.$$

Hence $o(u_1) = o(u) = md$ and $o(v_1) = o(v) = nd$. By (19) we have

$$\begin{aligned}
u_1 &= y^{-1}(v^{-1}y^2u^{-1})yy^{-2} = u^{-\beta}v^{-\delta},\\
v_1 &= x^2y(yxv^{-1})x^{-1} = x^2y^2v^{-\gamma} \overset{(19)}{=} uv^{1-\gamma}.
\end{aligned}$$

Then

$$\begin{aligned}
1 &= u_1^{md} = u^{-md\beta}v^{-md\delta} = v^{-md\delta},\\
1 &= v_1^{nd} = u^{nd}v^{nd(1-\gamma)} = u^{nd}.
\end{aligned}$$

Since $\delta \in \mathbb{Z}_{nd}^*$, we have $v^{md} = 1$ and $u^{nd} = 1$, and hence $n \mid m$ and $m \mid n$. Therefore $m = n$.

Now the relation $v^x = v^\gamma$ is mapped by θ to $v_1^{(xy)^{-1}} = v_1^\gamma$. Since

$$\begin{aligned}
v_1^{(xy)^{-1}} &= (uv^{1-\gamma})^{(xy)^{-1}} = u^{\alpha\beta}v^{\delta(\gamma-1)},\\
v_1^\gamma &= (uv^{1-\gamma})^\gamma = u^\gamma v^{\gamma(1-\gamma)} = u^\gamma v^{\gamma-1},
\end{aligned}$$

by equating the right-hand sides we obtain that $u^{\alpha\beta-\gamma} = v^{(1-\delta)(\gamma-1)}$. By Lemma 14, we have

$$u^{\alpha\beta-\gamma} \overset{(25)}{=} u^{\alpha-\beta-\gamma+1} \quad \text{and} \quad v^{(1-\delta)(\gamma-1)} = v^{(2-(\delta+1))(\gamma-1)} \overset{(26)}{=} v^{2(\gamma-1)} \overset{(23)}{=} u^{2(1-\beta)}.$$

It follows that $u^{\alpha-\beta-\gamma+1} = u^{2(1-\beta)}$, and consequently

$$\gamma \equiv \alpha+\beta-1 \pmod{md}. \tag{27}$$

Similarly, the relation $v^y = v^\delta$ is mapped to $v_1^x = v_1^\delta$. Since

$$v_1^x = (uv^{1-\gamma})^x = u^\alpha v^{\gamma-1} \quad \text{and} \quad v_1^\delta = (uv^{1-\gamma})^\delta = u^\delta v^{\delta(1-\gamma)},$$

we obtain $u^{\alpha-\delta} = v^{(\delta+1)(1-\gamma)} \overset{(26)}{=} 1$, and hence

$$\delta \equiv \alpha \pmod{md}. \tag{28}$$

Further, the relation $u^y = u^\beta$ is mapped to $u_1^x = u_1^\beta$. Since

$$u_1^x = (u^{-\beta}v^\delta)^x = u^{-\alpha\beta}v^{-\gamma\delta} \quad \text{and} \quad u_1^\beta = (u^{-\beta}v^{-\delta})^\beta = u^{-1}v^{-\beta\delta},$$

we obtain that

$$u^{\alpha\beta-1} = v^{(\beta-\gamma)\delta}.$$

By (25), (27) and (28) the above relation reduces to

$$u^{\alpha-\beta} \overset{(25)}{=} u^{\alpha\beta-1} = v^{(\beta-\gamma)\delta} \overset{(27),(28)}{=} v^{(1-\alpha)\alpha} = v^{\alpha-\alpha^2} = v^{\alpha-1}.$$

Recall that $\langle u \rangle \cap \langle v \rangle = \langle u^m \rangle = \langle v^m \rangle$, so we obtain

$$\alpha \equiv \beta \pmod{m}.$$

We then use the relation $u^m = v^{m\epsilon}$ to deduce that

$$v^{\alpha-1} = u^{\alpha-\beta} = (u^m)^{\frac{\alpha-\beta}{m}} = v^{(\alpha-\beta)\epsilon}.$$

Therefore $(\alpha-\beta)\epsilon \equiv \alpha - 1 \pmod{md}$.

Finally, the relation $u^m = v^{m\epsilon}$ is mapped by θ to $u_1^m = v_1^{m\epsilon}$. Since

$$\begin{aligned} u_1^m &= (u^{-\beta}v^{-\delta})^m = u^{-m\beta}v^{-m\delta}, \\ v_1^{m\epsilon} &= (uv^{1-\gamma})^{m\epsilon} = u^{m\epsilon}v^{m\epsilon(1-\gamma)}, \end{aligned}$$

we obtain that $u^{m(\epsilon+\beta)} = v^{m(-\delta+(\gamma-1)\epsilon)}$. Combining this with the relation $u^m = v^{m\epsilon}$ we have $v^{m(-\delta+(\gamma-1)\epsilon)} = v^{m\epsilon(\epsilon+\beta)}$, and hence

$$\epsilon(\epsilon+\beta) + \delta + (1-\gamma)\epsilon \equiv 0 \pmod{d}.$$

By (27) and (28) this reduces to $\epsilon^2 + (2-\alpha)\epsilon + \alpha \equiv 0 \pmod{d}$.

Conversely, it is straightforward to verify if the stated numerical conditions are fulfilled, then the above assignment extends to an automorphism of G, and hence the regular hypermap is triplly self-dual. $\square$

Recall that $\Omega_1 = \langle \omega_\tau, \omega_\pi \rangle$. So a regular hypermap is Ω_1-invariant if it is both symmetric and self-Petrie-dual. Combining Theorem 9, Theorem 12 and Theorem 13 we obtain the following corollary.

Corollary 16. *If an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering of the quaternion hypermap is Ω_1-invariant, then it is isomorphic to $\mathcal{H}(m, d, \alpha, \beta, \epsilon) = (G, x, y)$ where G is a group with a presentation*

$$\langle x, y \mid x^4 = u^{\alpha+1}v^{\beta-\alpha}, y^4 = u^{\beta-\alpha}v^{\alpha+1}, (xy)^4 = u^{\alpha+1}v^{\alpha+\beta}, u^{md} = v^{md} = 1,$$
$$u^x = u^\alpha, u^y = u^\beta, v^x = v^\beta, v^y = v^\alpha, u^m = v^{m\epsilon}, u = xy^{-1}xy, v = yx^{-1}yx\rangle,$$

and m and d are positive integers, $\alpha, \beta \in \mathbb{Z}_{md}^*$, $\epsilon \in \mathbb{Z}_d^*$ and they satisfy the following numerical conditions:

$$\begin{aligned}
&\alpha^2 \equiv 1 \pmod{md},\\
&\beta^2 \equiv 1 \pmod{md},\\
&(\beta-1)(\epsilon+1) \equiv 0 \pmod{md},\\
&\beta \equiv 1 \pmod{m},\\
&\beta \equiv -1 \pmod{d},\\
&\alpha \equiv \beta \pmod{d},\\
&\epsilon^2 \equiv 1 \pmod{d}.
\end{aligned}$$

A regular hypermap is *completely self-dual* if it is both symmetric and triplly self-dual. The following result classifies the coverings determined in Theorem 9 which are completely self-dual.

Corollary 17. *If an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering of the quaternion hypermap is completely self-dual, then it is isomorphic to one of the following regular hypermaps*

(i) $\mathcal{K}_1(m) = (G, x, y)$ *where*

$$G = \langle x, y \mid x^4 = u^2, y^4 = v^2, (xy)^4 = u^2v^2, u^m = v^m = 1,$$
$$u^x = u, u^y = u, v^x = v, v^y = v, u = xy^{-1}xy, v = yx^{-1}yx\rangle,$$

with $K = \langle u, v\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_m$ such that $G/K \cong \mathrm{Q}_8$.

(ii) $\mathcal{K}_2(m) = (G, x, y)$ *where*

$$G = \langle x, y \mid x^4 = u^2, y^4 = v^2, (xy)^4 = u^2v^2, u^{3m} = v^{3m} = 1,$$
$$u^x = u, u^y = u, v^x = v, v^y = v, u = xy^{-1}xy, v = yx^{-1}yx\rangle,$$

with $K = \langle u, v\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_{3m}$ such that $G/K \cong \mathrm{Q}_8$.

Proof. Since the covering is an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering of the quaternion hypermap, by Theorem 9, it is isomorphic to $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ for some integers $m,n,d,\alpha,\beta,\gamma,\delta$ and ϵ which fulfil the conditions (12) - (16). Since it is completely self-dual, it is both symmetric and triplly self-dual. By Theorem 12 and Theorem 15 we have $m=n$, $\beta\equiv\gamma \pmod{md}$ and $\gamma\equiv\alpha+\beta-1 \pmod{md}$. So $\alpha\equiv 1 \pmod{md}$, and hence the congruence $(\alpha-\beta)\epsilon\equiv\alpha-1 \pmod{md}$ in Theorem 15 reduces to $\beta\equiv 1 \pmod{md}$. It follows from Theorem 12 that $\gamma\equiv\delta\equiv 1 \pmod{md}$.

Moreover, upon substitution the congruence $\epsilon^2+(2-\alpha)\epsilon+\alpha\equiv 0 \pmod d$ in Theorem 15 reduces to $\epsilon^2+\epsilon+1\equiv 0 \pmod d$. By Theorem 12 we have $\epsilon^2\equiv 1 \pmod d$, so $\epsilon+2\equiv 0 \pmod d$, that is $\epsilon=d-2$. We have $\epsilon^2-1=d^2-4d+3$. Since $\epsilon^2\equiv 1 \pmod d$, we have $d\mid 3$. So either $d=1$ or $d=3$, giving the regular hypermaps in (i) and (ii) respectively. □

Recall that $\mho=\langle\omega_\tau,\omega_\pi,\omega_\theta\rangle$ and $\Omega=\langle\omega_\tau,\omega_\pi,\omega_\varsigma\rangle$. The following corollary determines the covering determined in Theorem 9 which are $\mho$-invariant.

Corollary 18. *If an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering of the quaternion hypermap is $\mho$-invariant, then it is isomorphic to $\mathcal{K}_1(m)$ given by Corollary 17. In particular, it is also Ω-invariant.*

Proof. Let $\mathcal{H}$ be an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering of the quaternion hypermap. If $\mathcal{H}$ is $\mho$-invariant, then it is symmetric, self-Petrie-dual and triplly self-dual. By Corollary 16 we have $\beta\equiv -1 \pmod d$. Recall that in the proof of Corollary 17 we have shown that $\beta=1$ and $d\mid 3$. So $d=1$, and hence $\mathcal{H}\cong\mathcal{K}_1(m)$. To show that $\mathcal{K}_1(m)=(G,x,y)$ is Ω-invariant, it suffices to verify that the assignment $\varsigma: x\mapsto y^{-1}, y\mapsto yx$ extends to an automorphism of G. This can be done in a similar manner as before, and we leave it as an exercise to the reader. □

7 Branched coverings

In this section we determine the coverings classified in Theorem 9 which are branched over hypervertices, over hyperedges or over hyperfaces.

Lemma 19. *Let $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ be an abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic covering over the quaternion hypermap classified in Theorem 9.*

(i) *If the covering is smooth at hypervertices, then*

$$\begin{aligned}&\alpha\equiv -1 \pmod m,\\&\delta\equiv 1 \pmod n,\\&\frac{\alpha+1}{m}\epsilon+\frac{\gamma-\delta}{n}\equiv 0 \pmod d.\end{aligned}$$

(ii) *If the covering is smooth at hyperedges, then*

$$\begin{aligned}
&\alpha \equiv 1 \pmod{m},\\
&\delta \equiv -1 \pmod{n},\\
&\frac{\beta-\alpha}{m}\epsilon + \frac{\delta+1}{n} \equiv 0 \pmod{d}.
\end{aligned}$$

(iii) *If the covering is smooth at hyperfaces, then*

$$\begin{aligned}
&\alpha \equiv -1 \pmod{m},\\
&\delta \equiv -1 \pmod{n},\\
&\frac{\alpha+1}{m}\epsilon + \frac{\gamma+\delta}{n} \equiv 0 \pmod{d}.
\end{aligned}$$

Proof. In the proof of Theorem 9 we have derived that

$$x^4 = u^{\alpha+1}v^{\gamma-\delta}, \quad y^4 = u^{\beta-\alpha}v^{\gamma-\delta} \quad \text{and} \quad (xy)^4 = u^{\alpha+1}v^{\gamma+\delta}.$$

(i) If the covering is smooth at hypervertices, then $x^4 = 1$, and hence

$$u^{\alpha+1} = v^{\delta-\gamma}.$$

Since $\langle u\rangle\cap\langle v\rangle = \langle u^m\rangle = \langle v^n\rangle$, we have $\alpha \equiv -1 \pmod{m}$ and $\delta \equiv \gamma \pmod{n}$. By (15) $\gamma \equiv 1 \pmod{n}$, so $\delta \equiv 1 \pmod{n}$. Using substitution $u^m = v^{n\epsilon}$ we deduce that $v^{\delta-\gamma} = u^{\alpha+1} = (u^m)^{\frac{\alpha+1}{m}} = v^{\frac{(\alpha+1)n}{m}\epsilon}$. Hence

$$\frac{\alpha+1}{m}\epsilon + \frac{\gamma-\delta}{n} \equiv 0 \pmod{d}.$$

(ii) If the covering is smooth at hyperedges, then $y^4 = 1$, and hence

$$u^{\alpha-\beta} = v^{\delta+1}.$$

Since $\langle u\rangle\cap\langle v\rangle = \langle u^m\rangle = \langle v^n\rangle$, we have $\alpha \equiv \beta \pmod{m}$ and $\delta \equiv -1 \pmod{n}$. By (15) $\beta \equiv 1 \pmod{n}$, so $\alpha \equiv 1 \pmod{n}$. Using substitution $u^m = v^{n\epsilon}$ we deduce that $v^{\delta+1} = u^{\alpha-\beta} = (u^m)^{\frac{\alpha-\beta}{m}} = v^{\frac{(\alpha-\beta)n}{m}\epsilon}$. Therefore

$$\frac{\beta-\alpha}{m}\epsilon + \frac{\delta+1}{n} \equiv 0 \pmod{d}.$$

(iii) If the covering is smooth at hyperfaces, then $(xy)^4 = 1$, and hence

$$u^{\alpha+1} = v^{-(\gamma+\delta)}.$$

Since $\langle u\rangle \cap \langle v\rangle = \langle u^m\rangle = \langle v^n\rangle$, we have $\alpha \equiv -1 \pmod m$ and $\delta \equiv -\gamma \pmod n$ By (15) $\gamma \equiv 1 \pmod n$, so $\delta \equiv -1 \pmod n$. Using substitution $u^m = v^{n\epsilon}$ we then deduce that $v^{-(\gamma+\delta)} = u^{\alpha+1} = (u^m)^{\frac{\alpha+1}{m}} = v^{\frac{(\alpha+1)n}{m}\epsilon}$. Hence

$$\frac{\alpha+1}{m}\epsilon + \frac{\gamma+\delta}{n} \equiv 0 \pmod d.$$

□

Theorem 20. *The abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic coverings $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ over the quaternion hypermap, branched over hyperfaces but smooth over both hypervertices and hyperedges, are summarized in Table 2.*

Table 2: Coverings which are branched over hyperfaces

Class	d	$(m,n,\alpha,\beta,\gamma,\delta,\epsilon)$	Type	Genus
i	odd	$(1,1,-1,-1,-1,-1,-1)$	$(4,4,4d)$	$2d$
ii	even	$(1,1,-1,-1,-1,-1,-1)$	$(4,4,2d)$	$2d-1$
iii	$d \geq 4$ even	$(1,1,-1,-1,-1,-1,\frac{d}{2}-1)$	$(4,4,2d)$	$2d-1$
iv	even	$(2,2,d-1,d-1,d-1,-1,-1)$	$(4,4,4d)$	$8d-3$
v		$(2,2,-1,-1,-1,-1,-1)$	$(4,4,4d)$	$8d-3$
vi	even	$(2,2,d-1,-1,-1,d-1,-1)$	$(4,4,4d)$	$8d-3$
vii	even	$(2,2,-1,d-1,d-1,d-1,-1)$	$(4,4,4d)$	$8d-3$
viii	1	$(1,2,1,1,1,1,1)$	$(4,4,4)$	3
ix	$d \geq 3$ odd	$(1,2,-1,-1,-1,-1,\frac{d-1}{2})$	$(4,4,4d)$	$4d-1$
x	1	$(2,1,1,1,1,1,1)$	$(4,4,4)$	3
xi	$d \geq 3$ odd	$(2,1,-1,-1,-1,-1,-2)$	$(4,4,4d)$	$4d-1$

Proof. Since the covering is smooth over hypervertices and hyperedges, by Theorem 9 and Lemma 19(i)-(ii), $1 \equiv \alpha \equiv -1 \pmod m$ and $1 \equiv \delta \equiv -1 \pmod n$, so $m \mid 2$ and $n \mid 2$. In what follows we distinguish four cases.

Case $(m = n = 1)$**.** By Theorem 9 we have $\gamma \equiv \alpha \pmod d$ and $\delta \equiv \beta \pmod d$. So in this case the numerical conditions in Theorem 9 and Lemma 19(i)-(ii) reduce to $\alpha^2 \equiv 1 \pmod d$, $\beta^2 \equiv 1 \pmod d$ and

$$(\beta-1)\epsilon + \alpha - 1 \equiv 0 \pmod d, \tag{29}$$

$$(\alpha+1)\epsilon + \alpha - \beta \equiv 0 \pmod d, \tag{30}$$

$$(\beta-\alpha)\epsilon + \beta + 1 \equiv 0 \pmod d. \tag{31}$$

We add (30) and (31) to obtain $(\beta+1)\epsilon + \alpha + 1 \equiv 0 \pmod d$. Subtracting (29) from this yields $2(\epsilon+1) \equiv 0 \pmod d$. Hence if d is odd, then $\epsilon = -1$,

while if d is even then the congruence reduces to

$$\epsilon + 1 \equiv 0 \pmod{\frac{d}{2}}.$$

Therefore, either $\epsilon = -1$ or $d \geq 4$ and $\epsilon = d/2 - 1$.

If $\epsilon = -1$, then using substitution in (30) and (31) we have $\alpha = \beta = -1$.

If $d \geq 4$ is even and $\epsilon = d/2 - 1$, then α and β are odd numbers. Using substitution in (29) and (30) we obtain that $\alpha = \beta = -1$.

Case $(m = n = 2)$**.** The numerical conditions in Theorem 9 and Lemma 19(i)-(ii) reduce to

$$\begin{aligned} \alpha^2 &\equiv 1 \pmod{2d}, & \gamma^2 &\equiv 1 \pmod{2d}, && (32)\\ \beta^2 &\equiv 1 \pmod{2d}, & \delta^2 &\equiv 1 \pmod{2d}, && (33)\\ \alpha &\equiv \gamma \pmod{d}, & \beta &\equiv \delta \pmod{d} && (34) \end{aligned}$$

and

$$\frac{\beta - 1}{2}\epsilon + \frac{\gamma - 1}{2} \equiv 0 \pmod{d}, \tag{35}$$

$$\frac{\alpha + 1}{2}\epsilon + \frac{\gamma - \delta}{2} \equiv 0 \pmod{d}, \tag{36}$$

$$\frac{\beta - \alpha}{2}\epsilon + \frac{\delta + 1}{2} \equiv 0 \pmod{d}. \tag{37}$$

We add (36) and (37) to obtain

$$\frac{\beta + 1}{2}\epsilon + \frac{\gamma + 1}{2} \equiv 0 \pmod{d}.$$

Subtracting (35) from this yields $\epsilon + 1 \equiv 0 \pmod{d}$. Hence $\epsilon = -1$. Using substitution in (35) and (36) we obtain that

$$\beta \equiv \gamma \pmod{2d}, \tag{38}$$

$$\gamma - \alpha - \delta - 1 \equiv 0 \pmod{2d}. \tag{39}$$

Recall that $\gamma \equiv \alpha \pmod{d}$. By (39) we have $\gamma - \alpha - \delta - 1 \equiv 0 \pmod{d}$, so $\delta \equiv -1 \pmod{d}$. Therefore either $\delta = -1$ or $\delta = d - 1$. In either case by (34) and (38) we have $\alpha, \beta, \gamma \in \{-1, d - 1\}$.

If $\delta = -1$, then (39) reduces to $\gamma \equiv \alpha \pmod{2d}$. Combining this with (34) and (38) we have either $\alpha = \beta = \gamma = d - 1$ or $\alpha = \beta = \gamma = -1$. In the first case by (32) the number d is even.

If $\delta = d-1$, then (39) reduces to $\gamma - \alpha \equiv d \pmod{2d}$. So either $\alpha = d-1$ and $\gamma = -1$, or $\alpha = -1$ and $\gamma = d - 1$. In the first case by (38) we have $\beta = -1$. Similarly in the latter case we have $\beta = d - 1$. In either case using substitution in (32) we have $d(d - 2) \equiv 0 \pmod{2d}$, and hence d is even.

Case $(m = 1$ and $n = 2)$**.** The numerical conditions in Theorem 9 and Lemma 19(i)-(ii) reduce to

$$\begin{aligned}
&\alpha^2 \equiv 1 \pmod{d}, && \delta^2 \equiv 1 \pmod{2d}, && (40)\\
&\beta^2 \equiv 1 \pmod{d}, && \gamma^2 \equiv 1 \pmod{2d}, && (41)\\
&\alpha \equiv \gamma \pmod{d}, && \beta \equiv \delta \pmod{d} && (42)
\end{aligned}$$

and

$$(\beta - 1)\epsilon + \frac{\gamma - 1}{2} \equiv 0 \pmod{d}, \tag{43}$$
$$(\alpha + 1)\epsilon + \frac{\gamma - \delta}{2} \equiv 0 \pmod{d}, \tag{44}$$
$$(\beta - \alpha)\epsilon + \frac{\delta + 1}{2} \equiv 0 \pmod{d}. \tag{45}$$

We add (44) and (45) to obtain

$$(\beta + 1)\epsilon + \frac{\gamma + 1}{2} \equiv 0 \pmod{d}.$$

Subtracting (43) from this yields $2\epsilon + 1 \equiv 0 \pmod{d}$. So d is an odd number. If $d = 1$, then $\alpha = \beta = \gamma = \delta = \epsilon = 1$, and if $d \geq 3$ is odd, then $\epsilon = (d-1)/2$. In the latter case the congruences (43) and (45) reduce to

$$(\beta - 1)(d - 1) + \gamma - 1 \equiv 0 \pmod{2d}, \tag{46}$$
$$(\beta - \alpha)(d - 1) + (\delta + 1) \equiv 0 \pmod{2d}, \tag{47}$$

which imply that

$$\begin{aligned}
&\beta \equiv \gamma \pmod{d}, && (48)\\
&\alpha - \beta + \delta + 1 \equiv 0 \pmod{d}.
\end{aligned}$$

Since $\beta \equiv \delta \pmod{d}$, the latter implies that $\alpha = -1$. Hence by (42) and (48) we have $\beta = -1$, and by (46) and (47) we have $\gamma = \delta = -1$.

Case $(m = 2$ and $n = 1)$**.** The numerical conditions in Theorem 9 and Lemma 19(i)-(ii) reduce to

$$\begin{aligned}
&\alpha^2 \equiv 1 \pmod{2d}, && \delta^2 \equiv 1 \pmod{d}, && (49)\\
&\beta^2 \equiv 1 \pmod{2d}, && \gamma^2 \equiv 1 \pmod{d}, && (50)\\
&\alpha \equiv \gamma \pmod{d}, && \beta \equiv \delta \pmod{d} && (51)
\end{aligned}$$

and

$$\frac{\beta-1}{2}\epsilon+\gamma-1\equiv 0 \pmod d, \tag{52}$$
$$\frac{\alpha+1}{2}\epsilon+\gamma-\delta\equiv 0 \pmod d, \tag{53}$$
$$\frac{\beta-\alpha}{2}\epsilon+\delta+1\equiv 0 \pmod d. \tag{54}$$

We add (53) and (54) to obtain

$$\frac{\beta+1}{2}\epsilon+\gamma+1\equiv 0 \pmod d.$$

Subtracting (52) from this yields $\epsilon+2\equiv 0 \pmod d$. So $\epsilon=-2$. Since $\epsilon\in\mathbb{Z}_d^*$, d must be odd. If $d=1$, then $\alpha=\beta=\gamma=\delta=\epsilon=1$, while if $d\geq 3$ is odd, then using substitution (52) and (53) are transformed into the form

$$\beta\equiv\gamma \pmod d,$$
$$\gamma-\delta-\alpha-1\equiv 0 \pmod d.$$

Combining these with (49) and (51) we obtain that $\alpha=\beta=\gamma=\delta=-1$.

By Corollary 10, we obtain the type and genus of the corresponding regular hypermaps in each case, as required. $\square$

Theorem 21. *The abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic coverings $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ over the quaternion hypermap, branched over hyperedges but smooth over both hypervertices and hyperfaces, are summarized in Table 3.*

Table 3: Regular coverings which are branched over hyperedges

Class	m	$(n,d,\alpha,\beta,\gamma,\delta,\epsilon)$	Type	Genus
i	odd	$(1,1,-1,1,1,1,1)$	$(4,4m,4)$	$2m$
ii	even	$(1,1,-1,1,1,1,1)$	$(4,2m,4)$	$2m-1$
iii		$(1,2,-1,1,1,1,1)$	$(4,4m,4)$	$4m-1$
iv	odd	$(2,1,-1,1,1,1,1)$	$(4,4m,4)$	$4m-1$
v	odd	$(2,1,-1,1,1,1,1)$	$(4,2m,4)$	$4m-3$

Proof. Since the covering $\mathcal{H}(m,n,d;\alpha,\beta,\gamma,\delta,\epsilon)$ is smooth over both hypervertices and hyperfaces, by Theorem 19(i) and (iii), the parameters satisfy

$$\frac{\alpha+1}{m}\epsilon+\frac{\gamma-\delta}{n}\equiv 0 \pmod d,$$
$$\frac{\alpha+1}{m}\epsilon+\frac{\gamma+\delta}{n}\equiv 0 \pmod d.$$

We subtract the first from the second and obtain

$$\frac{2\delta}{n} \equiv 0 \pmod{d}.$$

Since $\delta \in \mathbb{Z}_{nd}^*$, we have $nd \mid 2$. In what follows we distinguish three cases.

Case $(n = d = 1)$**.** It is clear that $\gamma = \delta = \epsilon = 1$. By reducing the numerical conditions we see that $\alpha = m - 1$ and $\beta = 1$.

Case $(n = 1$ and $d = 2)$**.** Clearly $\gamma = \delta = \epsilon = 1$. Upon substitution it is easy to solve the congruences to obtain that $\alpha = 2m - 1$ and $\beta = 1$.

Case $(n = 2$ and $d = 1)$**.** It is easy to obtain that $\alpha = m - 1$ and $\beta = \gamma = \delta = \epsilon = 1$.

By Corollary 10, we obtain the type and genus of the corresponding regular hypermaps in each case, as required. □

The proof of the following theorem is similar as the previous one, and we omit the details.

Theorem 22. *The abelian* $\{xy^{-1}xy, yx^{-1}yx\}$*-normalized bicyclic coverings* $\mathcal{H}(m, n, d; \alpha, \beta, \gamma, \delta, \epsilon)$ *over the quaternion hypermap, branched over hypervertices but smooth over both hyperedges and hyperfaces, are summarized in Table 4.*

Table 4: Coverings which are branched over hypervertices

Class	n	$(m, d, \alpha, \beta, \gamma, \delta, \epsilon)$	Type	Genus
i	odd	$(1, 1, 1, 1, 1, -1, 1)$	$(4n, 4, 4)$	$2n$
ii	even	$(1, 1, 1, 1, 1, -1, 1)$	$(2n, 4, 4)$	$2n - 1$
iii		$(1, 2, 1, 1, 1, -1, 1)$	$(4n, 4, 4)$	$4n - 1$
iv	odd	$(2, 1, 1, 1, 1, -1, 1)$	$(4n, 4, 4)$	$4n - 1$
v	even	$(2, 1, 1, 1, 1, -1, 1)$	$(2n, 4, 4)$	$4n - 3$

As a consequence of the above results we obtain the following classification of abelian $\{xy^{-1}xy, yx^{-1}yx\}$-normalized bicyclic coverings over the quaternion hypermap which are smooth simultaneously over hypervertices, hyperedges and hyperfaces.

Corollary 23. *A non-trivial abelian* $\{xy^{-1}xy, yx^{-1}yx\}$*-normalized bicyclic smooth covering over the quaternion hypermap is isomorphic to one of the following regular hypermaps:*

(i) $\mathcal{H}_1 = (G, x, y)$ *where*

$$G = \langle x, y \mid x^4 = y^4 = (xy)^4 = [x, y]^2 = 1, x^2 = y^2 \rangle.$$

(ii) $\mathcal{H}_2 = (G, x, y)$ *where*

$$G = \langle x, y \mid x^4 = y^4 = (xy)^4 = yx^{-1}yx = 1 \rangle.$$

(iii) $\mathcal{H}_3 = (G, x, y)$ *where*

$$G = \langle x, y \mid x^4 = y^4 = (xy)^4 = xy^{-1}xy = 1 \rangle.$$

(iv) $\mathcal{H}_4 = (G, x, y)$ *where*

$$G = \langle x, y \mid x^4 = y^4 = (xy)^4 = [x, y^2] = [y, x^2] = [x, y]^2 = 1 \rangle.$$

Remark 1. The regular hypermaps $\mathcal{H}_i$ $(i = 1, 2, 3)$, being 2-sheeted smooth coverings of the quaternion hypermap, all have type $(4, 4, 4)$ and genus 3. , The regular hypermap $\mathcal{H}_4$ has type $(4, 4, 4)$ and genus 5, arising as a 4-sheeted abelian regular smooth covering of the quaternion hypermap, as shown in Example 3. Checking Theorem 11, Theorem 13, Theorem 15 and Corollary 18 we find that $\mathcal{H}_1$ is reflexible, symmetric and self-Petrie-dual but not triplly self-dual, $\mathcal{H}_2$ and $\mathcal{H}_3$ are reflexible and self-Petrie-dual, but neither symmetric nor triplly self-dual, and $\mathcal{H}_4$ is totally symmetric. Moreover, all the groups in (i)-(iv) are 2-groups of class two, and in particular the group G in (iv) underlies a unique regular hypermap [16]. It can be verified that the underlying bipartite graph of $\mathcal{H}_1$ is $C_8^{(2)}$, a cycle of length 8 and multiplicity 2, and the underlying bipartite graphs of both $\mathcal{H}_2$ and $\mathcal{H}_3$ are the complete bipartite graph $K_{4,4}$, while that of $\mathcal{H}_4$ is the 4-dimensional hypercube.

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